

# THETA RANK, LEVELNESS, AND MATROID MINORS

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**ABSTRACT.** The Theta rank of a finite point configuration  $V$  is the maximal degree necessary for a sum-of-squares representation of a non-negative linear function on  $V$ . This is an important invariant for polynomial optimization that is in general hard to determine. We study the Theta rank and levelness, a related discrete-geometric invariant, for matroid base configurations. It is shown that the class of matroids with bounded Theta rank or levelness is closed under taking minors. This allows for a characterization of matroids with bounded Theta rank or levelness in terms of forbidden minors. We give the complete (finite) list of excluded minors for Theta-1 matroids which generalizes the well-known series-parallel graphs. Moreover, the class of Theta-1 matroids can be characterized in terms of the degree of generation of the vanishing ideal and in terms of the psd rank for the associated matroid base polytope. We further give a finite list of excluded minors for  $k$ -level graphs and matroids and we investigate the graphs of Theta rank 2.

## 1. INTRODUCTION

Let  $V$  be a configuration of finitely many points in  $\mathbb{R}^n$ . A linear function  $\ell(\mathbf{x}) = \delta - \langle c, \mathbf{x} \rangle$  which is non-negative on  $V$  is called  **$k$ -sos** with respect to  $V$  if there exist polynomials  $h_1, \dots, h_s \in \mathbb{R}[x_1, \dots, x_n]$  such that  $\deg h_i \leq k$  and

$$\ell(v) = h_1^2(v) + h_2^2(v) + \dots + h_s^2(v) \quad (1.1)$$

for all  $v \in V$ . The **Theta rank**  $\text{Th}(V)$  of  $V$  is the smallest  $k \geq 0$  such that every non-negative linear function is  $k$ -sos with respect to  $V$ . The Theta rank was introduced in [GPT10] as a measure for the ‘complexity’ of linear optimization over  $V$  using tools from polynomial optimization. If  $V$  is given as the solutions to a system of polynomial equations, then the size of a semidefinite program for the (exact) optimization of a linear function over  $V$  is of order  $O(n^{\text{Th}(V)})$ . For many practical applications, for example in combinatorial optimization, an algebraic description of  $V$  is readily available and the semidefinite programming approach is the method of choice. Clearly, situations with high Theta rank render the approach impractical. We are interested in

$$\mathcal{V}_k^{\text{Th}} := \{V \text{ point configuration} : \text{Th}(V) \leq k\}.$$

As  $V$  is finite and  $\ell(\mathbf{x})$  non-negative on  $V$ , we may interpolate  $\sqrt{\ell(\mathbf{x})}$  over  $V$  by a single polynomial which shows that  $\text{Th}(V) \leq |V| - 1$ . This, however, is a rather crude estimate as the 0/1-cube  $V = \{0, 1\}^n$  has Theta rank 1.

Let  $\ell(\mathbf{x})$  be a non-negative linear function. The subconfiguration  $V' = \{v \in V : \ell(v) = 0\}$  is called a **face** of  $V$  with supporting hyperplane  $H = \{\mathbf{x} \in \mathbb{R}^n : \ell(\mathbf{x}) = 0\}$ . If  $V' \neq V$  is inclusion-maximal, then  $V'$  is called a **facet** and  $H$  (and equivalently  $\ell(\mathbf{x})$ ) **facet-defining**. If  $V$  is a full-dimensional point configuration then  $H$  and  $\ell(\mathbf{x})$ , up to positive scaling, are unique. It follows from basic convexity that  $\text{Th}(V)$  is the smallest  $k$  such that all facet-defining linear

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functions  $\ell(\mathbf{x})$  are  $k$ -sos. A point configuration  $V$  is  $k$ -**level** if for every facet-defining hyperplane  $H$  there are  $k$  parallel hyperplanes  $H = H_1, H_2, \dots, H_k$  with

$$V \subseteq H_1 \cup H_2 \cup \dots \cup H_k.$$

Equivalently,  $V$  is  $k$ -level if every facet-defining linear function  $\ell(\mathbf{x})$  takes at most  $k$  distinct values on  $V$ . We say that a facet  $F$  is  $k$ -level if its facet-defining linear function  $\ell(\mathbf{x})$  takes exactly  $k$  distinct values on  $V$ . The **levelness**  $\text{Lev}(V)$  of  $V$  is the smallest  $k$  such that  $V$  is  $k$ -level. It is easy to see that  $\text{Th}(V) \leq \text{Lev}(V) - 1$ . Hence, the class  $\mathcal{V}_k^{\text{Lev}}$  of all  $k$ -level point configurations is a subclass of  $\mathcal{V}_{k-1}^{\text{Th}}$ . A main result of [GPT10] is the following characterization of  $\mathcal{V}_1^{\text{Th}}$ .

**Theorem 1.1** ([GPT10, Thm. 4.2]). *Let  $V$  be a finite point configuration. Then  $V$  has Theta rank 1 if and only if  $V$  is 2-level.*

For  $k \geq 2$ , it can be shown that  $\mathcal{V}_k^{\text{Lev}} \subsetneq \mathcal{V}_{k-1}^{\text{Th}}$ . The (convex) polytopes  $P = \text{conv}(V)$  for 2-level point configurations are very interesting. They arise in the study of extremal centrally-symmetric polytopes [SWZ09] as well as in statistics under the name of *compressed* polytopes [Sul06]. Every 2-level polytope is affinely isomorphic to a 0/1-polytope which gives them a combinatorial character. Nevertheless we lack a genuine understanding of such a family.

In this paper, we study the subclasses  $\mathcal{M}_k^{\text{Th}} \subset \mathcal{V}_k^{\text{Th}}$  of point configurations coming from the bases of *matroids*. We recall the notion of matroids and the associated geometric objects in Section 2. In particular, we show that the classes  $\mathcal{M}_k^{\text{Th}}$  are closed under taking *minors*. This, in principle, allows for a characterization of  $\mathcal{M}_k^{\text{Th}}$  in the form of forbidden sub-structures. In Section 3, we focus on the class  $\mathcal{M}_1^{\text{Th}}$  of matroids of Theta rank 1 or, equivalently, 2-level matroids. Our first main result is the following.

**Theorem 1.2.** *Let  $M = (E, \mathcal{B})$  be a matroid and  $V_M \subset \mathbb{R}^E$  the corresponding base configuration. The following are equivalent:*

- (i)  $V_M$  has Theta rank 1 or, equivalently, is 2-level;
- (ii)  $M$  has no minor isomorphic to  $M(K_4)$ ,  $\mathcal{W}^3$ ,  $Q_6$ , or  $P_6$ ;
- (iii)  $M$  can be constructed from uniform matroids by taking direct sums or 2-sums;
- (iv) The vanishing ideal  $I(V_M)$  is generated in degrees  $\leq 2$ ;
- (v) The base polytope  $P_M$  has minimal psd rank.

Part (ii) yields a complete and, in particular, finite list of excluded minors whereas (iii) gives a synthetic description of this class of matroids. The parts (iv) and (v) are proven in Section 4. The former states that 2-level matroids are precisely those matroids  $M$  for which the base configuration  $V_M$  is cut out by quadrics (Theorem 4.4). This contrasts the situation for general point configurations (Example 8). The psd rank of a polytope  $P$  is the smallest ‘size’ of a spectrahedron that linearly projects to  $P$ . The psd rank was studied in [GPT13, GRT13] and it was shown that the psd rank  $\text{Psd}(P)$  is at least  $\dim P + 1$ . Part (v) shows that the 2-level matroids are exactly those matroids for which the psd rank of the base polytope  $P_M = \text{conv}(V_M)$  is minimal. Again, this is in strong contrast to the psd rank of general polytopes.

In Section 5 we give a complete list of excluded minors for  $k$ -level graphs (Theorem 5.6). The classes of 3-level and 4-level graphs appear in works of Halin (see [Die90, Ch. 6]) and Oxley [Oxl89]. In particular, the wheel with 5 spokes  $W_5$  is shown to have Theta rank 3. Combined with results of Oxley [Oxl89], this yields a finite list of candidates for a complete characterization of Theta-2 graphs. Whereas the list of forbidden minors for graphs is always finite, this is generally not true for matroids. In Section 6 we show that  $k$ -levelness of matroids is characterized by finitely many excluded minors and we conjecture this to be true for matroids of Theta rank  $k$ .

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## 2. POINT CONFIGURATIONS AND MATROIDS

In this section we study properties of Theta rank and levelness related to the geometry of the point configuration. In particular, we investigate the behavior of these invariants under taking sub-configurations. We recall basic notions from matroid theory and associated point configurations and polytopes.

**2.1. Theta rank, levelness, and face-hereditary properties.** The definitions of levelness and Theta rank make only reference to the affine hull of the configuration  $V$  and thus neither depend on the embedding nor on a choice of coordinates. To have it on record we note the following basic property.

**Proposition 2.1.** *The levelness and the Theta rank of a point configuration are invariant under affine transformations.*

That this does not hold for (admissible) projective transformations is clear for the levelness and for the Theta rank follows from Theorem 1.1.

**Proposition 2.2.** *Let  $V_1 \subset \mathbb{R}^{d_1}$  and  $V_2 \subset \mathbb{R}^{d_2}$  be point configurations. Then the Theta rank satisfies  $\text{Th}(V_1 \times V_2) = \max(\text{Th}(V_1), \text{Th}(V_2))$ . The same is true for  $\text{Lev}(V_1 \times V_2)$ .*

*Proof.* A linear function  $\ell(\mathbf{x}, \mathbf{y})$  is facet defining for  $V_1 \times V_2$  if and only if  $\ell(\mathbf{x}, 0)$  is facet defining for  $V_1$  or  $\ell(0, \mathbf{y})$  is facet defining for  $V_2$ . Thus any representation (1.1) lifts to  $\mathbb{R}[\mathbf{x}, \mathbf{y}]$ .  $\square$

The Theta rank as well as the levelness of a point configuration are not monotone with respect to taking subconfigurations as can be seen by removing a single point from  $\{0, 1\}^d$ . However, it turns out that monotonicity holds for subconfigurations induced by supporting hyperplanes. Let us call a collection  $\mathcal{P}$  of point configurations **face-hereditary** if it is closed under taking faces. That is,  $V \cap H \in \mathcal{P}$  for any  $V \in \mathcal{P}$  and supporting hyperplane  $H$  for  $V$ .

**Lemma 2.3.** *The classes  $\mathcal{V}_k^{\text{Th}}$  and  $\mathcal{V}_k^{\text{Lev}}$  are face-hereditary.*

*Proof.* Let  $V \subset \mathbb{R}^d$  be a full-dimensional point configuration and  $H = \{p \in \mathbb{R}^d : g(p) = 0\}$  a supporting hyperplane such that the affine hull of  $V' := V \cap H$  has codimension 1. Let  $\ell(\mathbf{x})$  be facet-defining for  $V'$ . Observe that  $\ell(\mathbf{x})$  and  $\ell_\delta(\mathbf{x}) := \ell(\mathbf{x}) + \delta g(\mathbf{x})$  give the same linear function on  $V'$  for all  $\delta$ . For

$$\delta = \max \left\{ \frac{-\ell(v)}{g(v)} : v \in V \setminus V' \right\}$$

$\ell_\delta(\mathbf{x})$  is non-negative on  $V$ . Hence any representation (1.1) of  $\ell_\delta$  over  $V$  yields a representation for  $\ell$  over  $V'$ . Moreover, the levelness of  $\ell_\delta(\mathbf{x})$  gives an upper bound on the levelness of  $\ell(\mathbf{x})$ .  $\square$

It is interesting to note that these properties are not hereditary with respect to arbitrary hyperplanes. Indeed, consider the point configuration

$$V = (\{0, 1\}^n \times \{-1, 0, 1\}) \setminus \{\mathbf{0}\}$$

It can be easily seen that  $\text{Th}(V) = \text{Lev}(V) - 1 = 2$ . The hyperplane  $H = \{\mathbf{x} \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$  is not supporting and  $V' = V \cap H = \{0, 1\}^n \setminus \{\mathbf{0}\}$ . The linear function  $\ell(\mathbf{x}) = x_1 + \cdots + x_n - 1$  is facet-defining for  $V'$  with  $n$  levels. As for the Theta rank, any representation (1.1) yields a polynomial  $f(\mathbf{x}) = \ell(\mathbf{x}) - \sum_i h_i^2(\mathbf{x})$  of degree  $2k$  that vanishes on  $V'$  and  $f(\mathbf{0}) = -1 - \sum_i h_i^2(\mathbf{0}) < 0$ . For  $n > 4$ , the following proposition assures that  $\text{Th}(V') \geq 3$ .

**Proposition 2.4.** *Let  $V' = \{0, 1\}^n \setminus \{\mathbf{0}\}$  and  $f(\mathbf{x})$  a polynomial vanishing on  $V'$  and  $f(\mathbf{0}) \neq 0$ . Then  $\deg f \geq n$ .*

*Proof.* For a monomial  $\mathbf{x}^\alpha$ , let  $\tau = \{i : \alpha_i > 0\}$  be its support. Over the set of 0/1-points it follows that  $\mathbf{x}^\alpha$  and  $\mathbf{x}^\tau := \prod_{i \in \tau} x_i$  represent the same function. Hence, we can assume that  $f$  is of the form  $f(\mathbf{x}) = \sum_{\tau \subseteq [n]} c_\tau \mathbf{x}^\tau$  for some  $c_\tau \in \mathbb{R}$ ,  $\tau \subseteq [n]$ . Moreover  $c_\emptyset = f(\mathbf{0}) \neq 0$  and

without loss of generality we can assume  $c_\emptyset = 1$ . Any point  $v \in V'$  is of the form  $v = \mathbf{1}_\sigma$  for some  $\emptyset \neq \sigma \subseteq [n]$  and we calculate

$$0 = f(v) = \sum_{\emptyset \subseteq \tau \subseteq \sigma} c_\tau.$$

It follows that  $c_\tau$  satisfies the defining conditions of the *Möbius function* of the Boolean lattice and hence equals  $c_\tau = (-1)^{|\tau|}$  for all  $\tau \subseteq [n]$ . In particular  $c_{[n]} \neq 0$  which finishes the proof.  $\square$

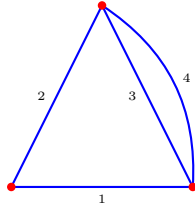
**2.2. Matroids and basis configurations.** We now introduce the combinatorial point configurations that are our main object of study. Matroids and their combinatorial theory are a vast subject and we refer the reader to the book by Oxley [Oxl11] for further information.

**Definition 2.5.** A **matroid** of rank  $k$  is a pair  $M = (E, \mathcal{B})$  consisting of a finite ground set  $E$  and a collection of bases  $\emptyset \neq \mathcal{B} \subseteq \binom{E}{k}$  satisfying the basis exchange axiom: for  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \setminus B_2$  there is  $y \in B_2 \setminus B_1$  such that  $(B_1 \setminus x) \cup y \in \mathcal{B}$ .

A set  $I \subseteq E$  is **independent** if  $I \subseteq B$  for some  $B \in \mathcal{B}$ . The **rank** of  $X$ , denoted by  $\text{rank}_M(X)$ , is the cardinality of the largest independent subset contained in  $X$ . The **circuits** of  $M$  are the inclusion-minimal dependent subsets. An element  $e$  is called a **loop** if  $\{e\}$  is a circuit. We say that  $e, f \in E$  are **parallel** if  $\{e, f\}$  is a circuit. A **parallel class**  $H \subseteq E$  is the equivalence class of elements parallel to each other. The class  $H$  is **non-trivial** if  $|H| > 1$ . A matroid is **simple** if it does not contain loops or parallel elements. A **flat** of a matroid is a set  $F \subseteq E$  such that  $\text{rank}(F) < \text{rank}(F \cup e)$  for all  $e \in E \setminus F$ .

A particular class of matroids that we will consider are the **graphic matroids**. To a graph  $G = (V, E)$  we associate the matroid  $M(G) = (E, \mathcal{B})$ . The bases are exactly the spanning forests of  $G$ . The running example for this section is the following.

**Example 1.** Let  $G$  be the graph



The graphic matroid  $M = M(G)$  has ground set  $E = \{1, 2, 3, 4\}$ ,  $\text{rank}(M) = 2$ , and bases

$$\mathcal{B}(G) = \{12, 13, 14, 23, 24\}.$$

The **dual matroid**  $M^*$  of the matroid  $M = (E, \mathcal{B})$  is the matroid defined by the pair  $(E, \mathcal{B}^*)$  where  $\mathcal{B}^* = \{E \setminus B : B \in \mathcal{B}\}$ . A **coloop** of  $M$  is an element which is a loop of  $M^*$ . Equivalently it is an element which appears in every basis of  $M$ .

If  $e \in E$  is not a coloop, we define the **deletion** as  $M \setminus e := (E \setminus e, \{B \in \mathcal{B} : e \notin B\})$ . If  $e$  is a coloop, then the bases of  $M \setminus e$  are  $\{B \setminus e : B \in \mathcal{B}\}$ . Dually, if  $e \in E$  is not a loop, we define the **contraction** as  $M/e := (E \setminus e, \{B \setminus e : e \in B \in \mathcal{B}\})$ . These operations can be extended to subsets  $X \subseteq E$  and we write  $M \setminus X$  and  $M/X$ , respectively. We also define the **restriction** of  $M$  to a subset  $X \subseteq E$  as  $M|_X := M \setminus (E \setminus X)$ . Note that  $(M \setminus X)^* = M^*/X$ . A **minor** of  $M$  is a matroid obtained from  $M$  by a sequence of deletion and contraction operations. The subclass of graphic matroids is closed under taking minors but not under taking duals.

To each matroid we associate a point configuration representing the set of bases. For a fixed ground set  $E$  let us write  $\mathbf{1}_X \in \{0, 1\}^E$  for the characteristic vector of  $X \subseteq E$ .

**Definition 2.6.** Let  $M = (E, \mathcal{B})$  be a matroid. The **base configuration** of  $M$  is the point configuration

$$V_M := \{\mathbf{1}_B : B \in \mathcal{B}\} \subset \mathbb{R}^E.$$

The **base polytope** of  $M$  is  $P_M := \text{conv}(V_M)$ .

The dual  $M^*$  is obtained by taking the complements of bases. The corresponding base configuration is thus

$$V_{M^*} = \mathbf{1} - V_M. \quad (2.1)$$

In particular,  $V_M$  and  $V_{M^*}$  are related by an affine transformation.

Observe that  $V_M$  is not a full-dimensional point configuration. Indeed,  $V_M$  is contained in the hyperplane  $\sum_{e \in E} x_e = \text{rank}(E)$ . In order to determine the dimension of  $V_M$  we need to consider the relations among elements of  $E$ :  $e_1, e_2 \in E$  are related if there exists a circuit of  $M$  containing both. This is an equivalence relation and the equivalence classes are called the **connected components** of  $M$ . Let us write  $c(M)$  for the number of connected components. The matroid  $M$  is **connected** if  $c(M) = 1$ .

Let  $M_1$  and  $M_2$  be matroids with disjoint ground sets  $E_1$  and  $E_2$ . The collection

$$\mathcal{B} := \{B_1 \cup B_2 : B_1 \in \mathcal{B}(M_1), B_2 \in \mathcal{B}(M_2)\}.$$

is the set of bases of a matroid on  $E_1 \cup E_2$ , called the **direct sum** of  $M_1$  and  $M_2$  and denoted by  $M_1 \oplus M_2$ . The corresponding base configuration is exactly the Cartesian product

$$V_{M_1 \oplus M_2} = V_{M_1} \times V_{M_2}. \quad (2.2)$$

If  $E_1, \dots, E_r \subseteq E$  are the connected components of  $M$ , then  $M = \bigoplus_i M|_{E_i}$ . Thus, showing that  $\dim V_M = |E| - 1$  if  $M$  is connected proves the following.

**Proposition 2.7.** *The smallest affine subspace containing  $V_M$  is of dimension  $|E| - c(M)$ .*

For a subset  $X \subseteq E$  let us write  $\ell_X(\mathbf{x}) = \sum_{e \in X} x_e$ . For  $A \subseteq E$  we then have  $\ell_X(\mathbf{1}_A) = |A \cap X|$ . Hence  $\text{rank}_M(X) = \max_{v \in P_M} \ell_X(v)$ . For  $X \subseteq E$  we define the supporting hyperplane

$$H_M(X) := \{\mathbf{x} \in \mathbb{R}^E : \ell_X(\mathbf{x}) = \text{rank}_M(X)\}.$$

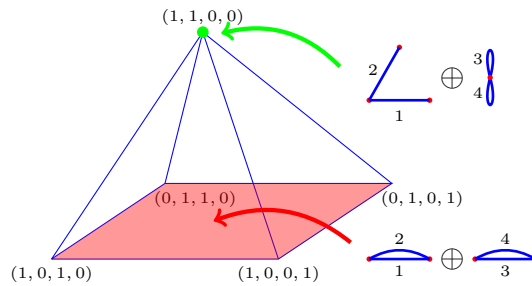
The corresponding faces of  $V_M$  (or equivalently of  $P_M$ ) are easy to describe.

**Proposition 2.8** ([Edm70]). *For a matroid  $M = (E, \mathcal{B})$  and a subset  $X \subset E$ , we have*

$$V_M \cap H_M(X) = V_{M|_X \oplus M/X} = V_{M|_X} \times V_{M/X}.$$

Let us illustrate this on our running example.

**Example 2** (continued). The graph given in Example 1 yields a connected matroid on 4 elements and hence a 3-dimensional base configuration. The corresponding base polytope is this:



The 5 bases correspond to the vertices of  $P_M$ . We considered the subset  $\{3, 4\}$  whose associated face  $M(G)|_{\{3,4\}} \times M(G)/\{3,4\}$  is the quadrilateral facet of the polytope, and the subset  $\{1, 2\}$  whose associated face  $M(G)|_{\{1,2\}} \times M(G)/\{1,2\}$  is the vertex  $(1, 1, 0, 0)$ .

We define the following families of matroids:

$$\mathcal{M}_k^{\text{Lev}} := \{M \text{ matroid} : \text{Lev}(V_M) \leq k\}, \text{ and}$$

$$\mathcal{M}_k^{\text{Th}} := \{M \text{ matroid} : \text{Th}(V_M) \leq k\}.$$

We will say that a matroid  $M$  is of Theta rank or level  $k$  if the corresponding base configuration  $V_M$  is. Now combining Proposition 2.8 with Lemma 2.3 proves the main theorem of this section.

**Theorem 2.9.** *The classes  $\mathcal{M}_k^{\text{Th}}$  and  $\mathcal{M}_k^{\text{Lev}}$  are closed under taking minors.*

*Proof.* Using Proposition 2.8 repeatedly on one-element sets shows that for every minor  $N$  of  $M$  there is a supporting hyperplane such that  $V_M \cap H$  is affinely isomorphic to  $V_N$ . Lemma 2.3 assures us that  $\text{Th}(V_N) \leq \text{Th}(V_M)$ .  $\square$

Let us analogously define the classes  $\mathcal{G}_k^{\text{Th}}$  and  $\mathcal{G}_k^{\text{Lev}}$  of graphic matroids of Theta rank and levelness bounded by  $k$ . These are also closed under taking minors and the Robertson–Seymour’s theorem ([RS04]) asserts that there is a finite list of excluded minors characterizing each class.

In the remainder of the section we will recall the facet-defining hyperplanes of  $V_M$  which will also show that *all* faces of  $V_M$  correspond to direct sums of minors. The facial structure of  $V_M$  has been of interest originally in combinatorial optimization [Edm70] (see also [Sch03, Ch. 40]) and later in geometric combinatorics and tropical geometry [AK06, FS05, Kim10].

**Theorem 2.10.** *Let  $M = (E, \mathcal{B})$  be a connected matroid. For every facet  $U \subset V_M$  there is a unique  $\emptyset \neq S \subset E$  such that  $U = V_M \cap H_M(S)$ . Conversely, a subset  $\emptyset \neq S \subset E$  gives rise to a facet if and only if*

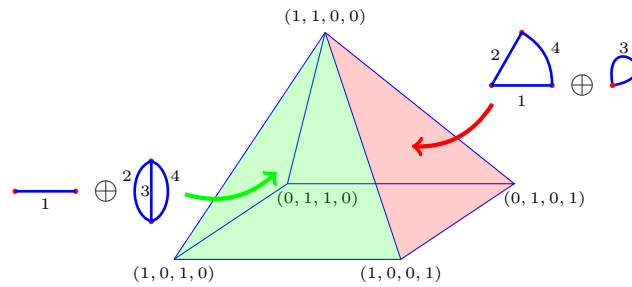
- (i)  $S$  is a flat such that  $M|_S$  as well as  $M/S$  are connected;
- (ii)  $S = E \setminus e$  for some  $e \in E$  such that  $M|_S$  as well as  $M/S$  are connected.

In [FS05] the subsets  $S$  in (i) were called **facets** and we stick to this name. In our study of the Theta rank and the levelness of base configurations, the following asserts that we will only need to consider facets. For brevity, a  $k$ -level facet refers to a facet whose corresponding facet is  $k$ -level.

**Proposition 2.11.** *Let  $M$  be a connected matroid and  $S = E \setminus e$ . Then  $\ell_S(\mathbf{x})$  takes 2 values on  $V_M$  and hence is 1-sos.*

*Proof.* Let  $r$  be the rank of  $M$ . Restricted to the affine hull of  $V_M$ , we have that  $\ell_S(\mathbf{x})$  and  $r - \mathbf{x}_e$  induce the same linear function. As  $V_M$  is a 0/1-configuration, it follows that  $\ell_S(\mathbf{x})$  takes the 2 values  $r$  and  $r - 1$  on  $V_M$ .  $\square$

**Example 3.** The facets of the running example are four triangles and one square. The four triangles correspond to the two sets  $\{1, 2, 4\}$ ,  $\{1, 2, 3\}$  of cardinality  $|E| - 1$  and the two facets  $\{2\}$ ,  $\{1\}$ , while the square corresponds to the facet  $\{3, 4\}$ . We have already described in the previous example the square facet. In the picture we highlight two triangular facets, the first one (green) corresponding to the facet  $\{1\}$ , the second one (red) to the set  $\{1, 2, 4\}$ .



A seemingly trivial but useful class of matroids is given by the **uniform matroids**  $U_{n,k}$  for  $0 \leq k \leq n$  given on ground set  $E = \{1, \dots, n\}$  and bases  $\mathcal{B}(U_{n,k}) = \{B \subseteq E : |B| = k\}$ .

**Proposition 2.12.** *Uniform matroids are 2-level and hence have Theta rank 1.*

*Proof.* The base polytope of  $U_{n,k}$  is also known as the  $(n, k)$ -hypersimplex and is given by

$$P_{U_{n,k}} = \text{conv}\{\mathbf{1}_B : B \subseteq E, |B| = k\} = \left\{ \mathbf{x} \in \mathbb{R}^E : 0 \leq x_e \leq 1, \sum_e x_e = k \right\}.$$

The facet-defining linear functions are among the functions  $\{\pm \ell_{\{e\}}(\mathbf{x}) = \pm x_e : e \in E\}$  which can take only two different values on 0/1-points.  $\square$



## 3. 2-LEVEL MATROIDS

In this section we investigate the excluded minors for the class of 2-level matroids and, by Theorem 1.1, equivalently the matroids of Theta rank 1. In this case we can give the complete and in particular finite list of forbidden minors. We start by showing that we can exclude matroids with few elements and of small rank.

**Proposition 3.1.** *Let  $M = (E, \mathcal{B})$  be a matroid. If  $\text{rank}(M) \leq 2$  or  $|E| \leq 5$ , then  $M$  is 2-level.*

*Proof.* The case  $\text{rank}(M) = 1$  is trivial since there is no proper facet. On the other hand, if  $\text{rank}(M) = 2$  the proper facets are necessarily facets of rank 1. The linear function  $\ell_F(\mathbf{x})$  for any such facet  $F$  only takes values in  $\{0, 1\}$  and thus is 2-level. By (2.1) and Proposition 2.1,  $M$  and  $M^*$  have the same Theta rank and levelness. If  $|E| \leq 5$ , then either  $M$  or  $M^*$  is of rank  $\leq 2$ .  $\square$

A first example of a matroid of levelness  $\geq 3$  is given by the graphic matroid associated to the complete graph  $K_4$ .

**Proposition 3.2.** *The graphic matroid  $M(K_4)$  is 3-level.*

*Proof.* Let  $F = \{1, 2, 3\}$  be the flat corresponding to the labelled example shown below. Both the contraction of  $F$  and the restriction to  $F$  are connected (or biconnected on the level of graphs) and thus  $F$  is a facet with  $\ell_F(\mathbf{x}) = x_1 + x_2 + x_3$ . The spanning trees  $B_1 = \{1, 5, 6\}$  and  $B_2 = \{4, 5, 6\}$  satisfy  $|F \cap B_2| < |F \cap B_1| < \text{rank}(F)$  which shows that  $M(K_4)$  is at least 3-level. To see that  $M(K_4)$  is at most 3-level we notice that every proper facet  $F$  has rank smaller or equal than  $\text{rank}(M(K_4)) - 1 = 2$  and hence  $\ell_F(\mathbf{x})$  can take at most three different values.  $\square$

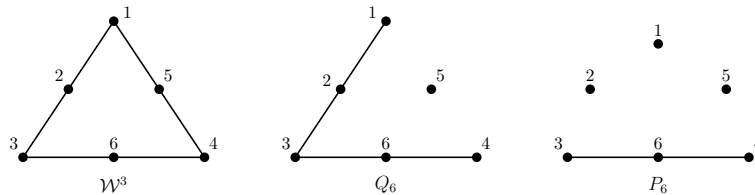
Before analyzing other matroids we quickly recall a **geometric representation** of certain matroids of rank 3: The idea is to draw a diagram in the plane whose points correspond to the elements of the ground set. Any subset of 3 elements constitute a basis unless they are contained in a depicted line.

**Example 4.** Let us consider the graph  $K_4$  and its geometric representation as a matroid:



Thus the geometric representation consists only of the four lines associated to the 3-circuits of  $K_4$ .

Starting from the geometric representation of  $M(K_4)$  we define three new matroids by removing one, two or three lines of the representation and we call them respectively  $\mathcal{W}^3$ ,  $Q_6$  and  $P_6$ . None of these matroids is graphic, but we can easily draw their geometric representations:



**Proposition 3.3.** *The matroids  $\mathcal{W}^3$ ,  $Q_6$ , and  $P_6$  are 3-level.*

*Proof.* Let  $M$  be any of the three given matroids and consider  $F = \{1, 2, 3\}$ . It is easy to check that  $M|_F \cong U_{3,2}$  and  $M/F \cong U_{3,1}$  which marks  $F$  as a facet. The vertices of the matroid

polytope associated to the bases  $\{4, 5, 6\}$ ,  $\{1, 4, 6\}$ ,  $\{1, 2, 6\}$  lie on distinct hyperplanes parallel to  $H_M(F) = \{\ell_F(\mathbf{x}) = \text{rank}_M(F)\}$ . Therefore the matroids are at least 3-level. Since  $\text{rank}(M) = 3$ , we can use the same argument as in the proof of Proposition 3.2.  $\square$

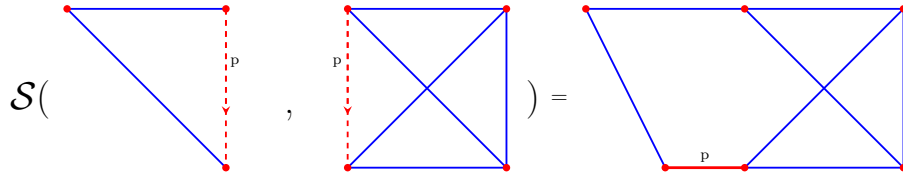
The list of excluded minors for  $\mathcal{M}_2^{\text{Lev}}$  so far includes  $M(K_4)$ ,  $\mathcal{W}^3$ ,  $Q_6$ , and  $P_6$ . To show that this list is complete, we will approach the problem from the constructive side and consider how to synthesize 2-level matroids. We already saw that  $\mathcal{M}_2^{\text{Lev}}$  is closed under taking direct sums. We will now consider three more operations that retain levelness. Let  $M_1 = (E_1, \mathcal{B}_1)$  and  $M_2 = (E_2, \mathcal{B}_2)$  be matroids such that  $\{p\} = E_1 \cap E_2$ . We call  $p$  a **base point**. If  $p$  is not a coloop of both, then we define the **series connection**  $\mathcal{S}(M_1, M_2)$  with respect to  $p$  as the matroid on ground set  $E_1 \cup E_2$  and with bases

$$\mathcal{B} = \{B_1 \cup B_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2, B_1 \cap B_2 = \emptyset\}.$$

We also define the **parallel connection** with respect to  $p$  as the matroid  $\mathcal{S}(M_1^*, M_2^*)^*$  provided  $p$  is not a loop of both. Notice that  $\mathcal{S}(M_1, M_2)$  contains both  $M_1$  and  $M_2$  as a minor.

The operations of series and parallel connection, introduced by Brylawski [Bry71], are inspired by the well-known series and parallel operations on graphs. The following example illustrates the construction in the graphic case.

**Example 5.** Let us consider again the two graphic matroids  $U_{3,2}$  and  $M(K_4)$ . Their series connection is the following graph:



An extensive treatment of these two operations is given in [Oxl11, Sect. 7.1]. We focus here on the geometric properties from which many combinatorial consequences can be deduced. For the following result, we write  $E_1 \uplus E_2 = (E_1 \cup E_2 \cup \{p_1, p_2\}) \setminus \{p\}$  for the *disjoint union* of  $E_1$  and  $E_2$ .

**Lemma 3.4.** *Let  $M_1 = (E_1, \mathcal{B}_1)$  and  $M_2 = (E_2, \mathcal{B}_2)$  be matroids with  $\{p\} = E_1 \cap E_2$  not a coloop of both. Then the base polytope  $P_{\mathcal{S}}$  of the series connection  $\mathcal{S} = \mathcal{S}(M_1, M_2)$  is linearly isomorphic to*

$$(P_{M_1} \times P_{M_2}) \cap \{\mathbf{x} \in \mathbb{R}^{E_1 \uplus E_2} : x_{p_1} + x_{p_2} \leq 1\}.$$

*Proof.* It is clear that the base configuration  $V_{\mathcal{S}}$  is isomorphic to

$$V' = (V_{M_1} \times V_{M_2}) \cap \{\mathbf{x} \in \mathbb{R}^{E_1 \uplus E_2} : x_{p_1} + x_{p_2} \leq 1\}$$

under the linear map  $\pi : \mathbb{R}^{E_1 \uplus E_2} \rightarrow \mathbb{R}^{E_1 \cup E_2}$  given by  $\pi(\mathbf{1}_{p_1}) = \pi(\mathbf{1}_{p_2}) = \mathbf{1}_p$  and  $\pi(\mathbf{1}_e) = \mathbf{1}_e$  otherwise. Indeed, let  $r_i = \text{rank}(M_i)$ , then a linear inverse is given by  $s : \mathbb{R}^{E_1 \cup E_2} \rightarrow \mathbb{R}^{E_1 \uplus E_2}$  with  $s(\mathbf{x})_{p_i} = r_i - \ell_{E_i}(\mathbf{x})$  for  $i = 1, 2$  and the identity otherwise.

It is therefore sufficient to show that the vertices of

$$P' = (P_{M_1} \times P_{M_2}) \cap \{\mathbf{x} \in \mathbb{R}^{E_1 \uplus E_2} : x_{p_1} + x_{p_2} \leq 1\}.$$

are exactly the points in  $V'$ . Clearly  $V'$  is a subset of the vertices and any additional vertex of  $P'$  would be the intersection of the relative interior of an edge of  $P_{M_1} \times P_{M_2}$  with the hyperplane  $H = \{\mathbf{x} : x_{p_1} + x_{p_2} = 1\}$ . However, every edge of  $P_{M_1} \times P_{M_2}$  is parallel to some  $\mathbf{1}_e - \mathbf{1}_f$  for  $e, f \in E_1$  or  $e, f \in E_2$ . Thus every edge of  $P_{M_1} \times P_{M_2}$  can meet  $H$  only in one of its endpoints which proves the claim.  $\square$

It is interesting to note that the operation that related  $P_{M_1}$  and  $P_{M_2}$  to  $P_{\mathcal{S}(M_1, M_2)}$  is exactly a *subdirect product* in the sense of McMullen [McM76]. From the description of  $P_{\mathcal{S}(M_1, M_2)}$  we instantly get information about the Theta rank and levelness of the series and parallel connection.



**Corollary 3.5.** *Let  $\mathcal{S} = \mathcal{S}(M_1, M_2)$  be the series connection of matroids  $M_1$  and  $M_2$ . Then*

$$\text{Th}(\mathcal{S}) = \max(\text{Th}(M_1), \text{Th}(M_2)).$$

*The same holds true for the parallel connection as well as for the levelness.*

*Proof.* Lemma 3.4 shows that the facet-defining linear functions of  $P_{\mathcal{S}}$  are among those of  $P_{M_1} \times P_{M_2}$  and  $\ell(\mathbf{x}) = x_{p_1} + x_{p_2}$ . However, by the characterization of the bases of  $\mathcal{S}$ ,  $\ell(\mathbf{x})$  can take only values in  $\{0, 1\}$ . Hence,  $\text{Th}(V_{\mathcal{S}}) = \text{Th}(V_{M_1} \times V_{M_2})$  and Proposition 2.2 finishes the proof.  $\square$

**Corollary 3.6.** *The classes  $\mathcal{M}_k^{\text{Th}}$  and  $\mathcal{M}_k^{\text{Lev}}$  are closed under taking series and parallel connections.*

The most important operation that we will need is derived from the series connection. Let  $M_1 = (E_1, \mathcal{B}_1)$  and  $M_2 = (E_2, \mathcal{B}_2)$  be matroids with  $E_1 \cap E_2 = \{p\}$ . If  $p$  is not a loop nor a coloop for neither  $M_1$  nor  $M_2$ , then we define the **2-sum**

$$M_1 \oplus_2 M_2 := \mathcal{S}(M_1, M_2)/p.$$

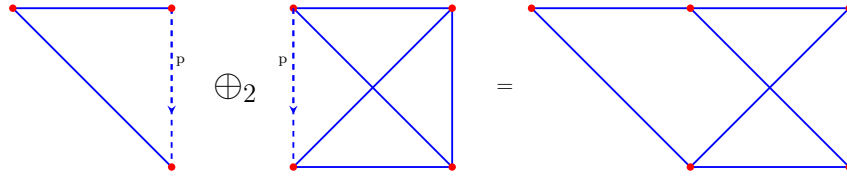
This is the matroid on the ground set  $E = (E_1 \cup E_2) \setminus p$  and with bases

$$\mathcal{B} := \{B_1 \cup B_2 \setminus p : B_1 \in \mathcal{B}_M, B_2 \in \mathcal{B}_N, p \in B_1 \triangle B_2\}$$

where  $B_1 \triangle B_2$  is the symmetric difference.

The 2-sum is an associative operation for matroids which defines, by analogy to the direct sum, the 3-connectedness: a connected matroid  $M$  is **3-connected** if and only if it cannot be written as a 2-sum of two matroids each with fewer elements than  $M$ .

**Example 6.** Let us consider the 2-sum of a matroid  $U_{3,2} \oplus_2 M(K_4)$ : both matroids are graphic, therefore we can illustrate the operation for the corresponding graphs.



To perform the 2-sum we select an element for each matroid, while in the picture it looks like we also need to orient the chosen element. This is the case only because we are drawing an embedding of a graphic matroids; in fact the structure given by the vertices is forgotten when we look at the matroid. Whitney's 2-Isomorphism Theorem [Oxl11, Thm. 5.3.1] clarifies that the matroid structure does not depend on the orientation we decide for the chosen elements.

We will need the following two properties of 2-sums.

**Lemma 3.7** ([CO03, Lem. 2.3]). *Let  $M$  be a 3-connected matroid having no minor isomorphic to any of  $M(K_4)$ ,  $\mathcal{W}^3$ ,  $Q_6$ ,  $P_6$ . Then  $M$  is uniform.*

**Lemma 3.8** ([Oxl11, Thm. 8.3.1]). *Every matroid that is not 3-connected can be constructed from 3-connected proper minors of itself by a sequence of direct sums and 2-sums.*

We can finally give a complete characterization of the class  $\mathcal{M}_2^{\text{Lev}} = \mathcal{M}_1^{\text{Th}}$ .

**Theorem 3.9.** *For a matroid  $M$  the following are equivalent.*

- (i)  $M$  has Theta rank 1.
- (ii)  $M$  is 2-level.
- (iii)  $M$  has no minor isomorphic to  $M(K_4)$ ,  $\mathcal{W}^3$ ,  $Q_6$ , or  $P_6$ .
- (iv)  $M$  can be constructed from uniform matroids by taking direct or 2-sums.

*Proof.* (i)  $\Rightarrow$  (ii) is just Theorem 1.1. (ii)  $\Rightarrow$  (iii) follows from Theorem 2.9 and Proposition 3.3. Let  $M$  be a matroid satisfying (iii). If  $M$  is 3-connected, then  $M$  is uniform by Lemma 3.7. If  $M$  is not 3-connected, then Lemma 3.8 shows that it satisfies (iv). Finally, uniform matroids have

Theta rank 1 by Proposition 2.12. Theta rank  $\leq k$  is retained by series connection (Corollary 3.5) and, by definition, also by the 2-sum.  $\square$

**Example 7.** If we look at the family of 2-level graphic matroids, the only excluded minor is the graph  $K_4$ . The class of graphs which do not contain  $K_4$  as a minor is the well-known class of **series-parallel graphs**  $\mathcal{G}_{\text{SP}}$ . The theorem implies  $\mathcal{G}_2^{\text{Lev}} = \mathcal{G}_{\text{SP}}$ .

There are other point configurations that are naturally associated to a matroid  $M$ , most notably the configuration  $D_M = \{\mathbf{1}_X : X \subseteq E \text{ dependent}\}$ . For *binary* matroids, the associated polytope (up to translation and scaling) is called the *cycle polytope*. The practical relevance stems from the situation where  $M = M(G)^*$  for some graph  $G$ . In this case,  $D_M$  represents the collection of cuts in  $G$  which are important in combinatorial optimization. The Theta rank of  $D_M$  has been studied in [GLPT12]. In particular, the paper gives a characterization of binary matroids with  $\text{Th}(D_M) = 1$  in terms of forbidden minors with some additional conditions on the cocircuits. The situation is slightly different as the Theta rank of circuit configurations is monotone with respect to deletion minors but not necessarily with respect to contraction minors. The characterization of 2-level cut polytopes has been also obtained by Sullivant [Sul06].

#### 4. GENERATION AND PSD RANK

In this section we study two further face-hereditary properties of point configurations that are intimately related to Theta-1 configurations.

**4.1. Degree of generation.** For a point configuration  $V \subset \mathbb{R}^d$ , the **vanishing ideal** of  $V$  is

$$I(V) := \{f(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_d] : f(v) = 0 \text{ for all } v \in V\}.$$

We say that  $V$  is of **degree**  $\leq k$  if the ideal  $I(V)$  has some set of generators of degree  $\leq k$ . We write  $\text{Gen}(V) = k$  for the maximal degree in any minimal generating set for  $I(V)$ . We define

$$\mathcal{V}_k^{\text{Gen}} := \{V \text{ point configuration} : \text{Gen}(V) \leq k\}$$

It is clear that  $\text{Gen}(V)$  is an affine invariant and, since all point configurations are finite, we get

**Proposition 4.1.** *The class  $\mathcal{V}_k^{\text{Gen}}$  is face-hereditary.*

*Proof.* Let  $H = \{p : \ell(p) = 0\}$  be a supporting hyperplane for  $V$ . The vanishing ideal of  $V' = V \cap H$  is the ideal generated by  $I(V)$  and  $\ell(\mathbf{x})$ . Since  $\ell(\mathbf{x})$  is linear, this then shows that  $\text{Gen}(V') \leq \text{Gen}(V)$ .  $\square$

The relation to point configurations of Theta rank 1 is given by the following proposition which is implicit in [GPT10].

**Proposition 4.2.** *If  $V \subset \mathbb{R}^d$  be a point configuration of Theta rank 1, then  $\text{Gen}(V) \leq 2$ .*

*Proof.* From Theorem 1.1 we infer that the points  $V$  are in convex position and the polytope  $P = \text{conv}(V)$  is 2-level. We may assume that the configuration is spanning and hence up to affine equivalence, the polytope is given by

$$P = \left\{ p \in \mathbb{R}^d : \begin{array}{ll} 0 \leq p_i \leq 1 & \text{for } i = 1, \dots, d \\ \delta_j^- \leq \ell_j(p) \leq \delta_j^+ & \text{for } j = 1, \dots, n \end{array} \right\}$$

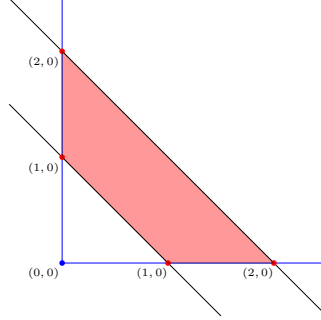
for unique linear functions  $\ell_j(\mathbf{x})$  and  $\delta_j^- < \delta_j^+$ . In particular,  $V \subset \{0, 1\}^d$ . We claim that  $I(V)$  is generated by the quadrics

$$x_i(x_i - 1) \quad \text{for } 1 \leq i \leq d, \quad (\ell_j(\mathbf{x}) - \delta_j^-)(\ell_j(\mathbf{x}) - \delta_j^+) \quad \text{for } 1 \leq j \leq n.$$

The vanishing locus  $U$  is a smooth and real subset of  $\{0, 1\}^d$ . Thus, the polynomials span a real radical ideal. Now, every vertex  $v \in V \subseteq \{0, 1\}^d$  satisfies  $\ell_j(v) = \delta_j^\pm$ . Hence  $V \subseteq U$ . Conversely, every  $u \in U$  is a vertex of  $P$  and hence  $U \subseteq V$ .  $\square$

The following example illustrates the fact that degree of generation is invariant under projective transformations while Theta rank is not.

**Example 8.** To see that generation in degrees  $\leq 2$  is necessary for Theta rank 1 but not sufficient, consider the planar point configuration  $V = \{(1, 0), (0, 1), (2, 0), (0, 2)\}$ . The configuration is clearly not 2-level and hence not Theta 1, however the vanishing ideal  $I(V)$  is generated by  $x_1x_2$  and  $(x_1 + x_2 - 1)(x_1 + x_2 - 2)$  which implies  $\text{Gen}(V) \leq 2$ .



The vanishing ideals of base configurations are easy to write down explicitly.

**Proposition 4.3.** *Let  $M = (E, \mathcal{B})$  be a matroid of rank  $r$ . The vanishing ideal for  $V_M$  is generated by*

$$x_e^2 - x_e \text{ for all } e \in E, \quad \ell_E(\mathbf{x}) - r, \quad \mathbf{x}^C \text{ for all circuits } C \subset E.$$

*Proof.* Any solution to the first two sets of equations is of the form  $\mathbf{1}_B$  for some  $B \subseteq E$  with  $|B| = r$ . For the last set of equations, we note that  $(\mathbf{1}_B)^C = 0$  for all circuits  $C$  if and only if  $B$  does not contain a circuit. This is equivalent to  $B \in \mathcal{B}$ . Arguments similar to those used in the proof of Proposition 4.2 show that the polynomials generate a real radical ideal.  $\square$

Let us write  $\mathcal{M}_k^{\text{Gen}}$  for the class of matroids  $M$  with  $\text{Gen}(V_M) \leq k$ . The previous proposition is a little deceiving in the sense that it suggests a direct connection between the size of circuits and the degree of generation. This is not quite true. Indeed, let  $G = K_4 \setminus e$  be the complete graph on 4 vertices minus an edge. Then  $M(G)$  has a circuit of cardinality 4 but  $M(G) \in \mathcal{M}_2^{\text{lev}} \subseteq \mathcal{M}_2^{\text{Gen}}$  by Theorem 3.9 and Proposition 4.2. The main result of this section is that for base configurations the condition of Proposition 4.2 is also sufficient.

**Theorem 4.4.** *Let  $M$  be a matroid. Then  $V_M$  is Theta 1 if and only if  $\text{Gen}(V_M) \leq 2$ .*

*Proof.* From Proposition 4.2 we already know that  $\mathcal{M}_1^{\text{Th}} \subseteq \mathcal{M}_2^{\text{Gen}}$ . Now, if  $M \in \mathcal{M}_2^{\text{Gen}} \setminus \mathcal{M}_1^{\text{Th}}$ , then  $M$  has a minor isomorphic to  $M(K_4)$ ,  $P_6$ ,  $Q_6$ , or  $\mathcal{W}^3$ . Since  $\mathcal{M}_2^{\text{Gen}}$  is closed under taking minors, the following proposition yields a contradiction.  $\square$

**Proposition 4.5.**  *$M(K_4)$ ,  $\mathcal{W}^3$ ,  $Q_6$ , and  $P_6$  are not in  $\mathcal{M}_2^{\text{Gen}}$ .*

*Proof.* For a point configuration  $V \subset \mathbb{R}^n$ , let  $I \subset \mathbb{R}[x_1, \dots, x_n]$  be its vanishing ideal. If  $I$  is generated in degrees  $\leq k$ , then so is any Gröbner basis of  $I$  with respect to a degree-compatible term order. The claim can now be verified by, for example, using the software *Macaulay2* [GS].  $\square$

**Psd rank and minimality.** Let  $\mathcal{S}^m \subset \mathbb{R}^{m \times m}$  be the vector space of symmetric  $m \times m$  matrices. The **psd cone** is the closed convex cone  $\mathcal{S}_+^m = \{A \in \mathcal{S}^m : A \text{ positive semidefinite}\}$ .

**Definition 4.6.** A polytope  $P \subset \mathbb{R}^d$  has a **psd-lift** of size  $m$  if there is a linear subspace  $L \subset \mathcal{S}^m$  and a linear projection  $\pi : \mathcal{S}^m \rightarrow \mathbb{R}^d$  such that  $P = \pi(\mathcal{S}_+^m \cap L)$ . The **psd rank**  $\text{Psd}(P)$  is the size of a smallest psd-lift.

Psd-lifts together with lifts for more general cones were introduced by Gouveia, Parrilo, and Thomas [GPT13] as natural generalization of *polyhedral lifts* or *extended formulations*. Let us define  $\mathcal{V}_k^{\text{Psd}}$  as the class of point configurations  $V$  in convex position such that  $\text{conv}(V)$  has a psd-lift of size  $\leq k$ . In [GRT13] it was shown that for a  $d$ -dimensional polytope  $P$  the psd rank is always  $\geq d + 1$ . A polytope  $P$  is called **psd-minimal** if  $\text{Psd}(P) = \dim P + 1$ . We write  $\mathcal{V}_{\min}^{\text{Psd}}$  for the class of psd-minimal (convex position) point configurations.

**Proposition 4.7.** *The classes  $\mathcal{V}_k^{\text{Psd}}$  and  $\mathcal{V}_{\min}^{\text{Psd}}$  are face-hereditary.*

*Proof.* Let  $V \in \mathcal{V}_k^{\text{Psd}}$  and let  $(L, \pi)$  be a psd-lift of  $P = \text{conv}(V)$ . For a supporting hyperplane  $H$  we observe that  $(L \cap \pi^{-1}(H), \pi)$  is a psd-lift of  $P \cap H$  of size  $m \leq k$ .

Let  $P$  be psd-minimal and let  $F = P \cap H$  a face of dimension  $\dim F = \dim P - 1$ . If  $F$  is not psd-minimal, then by [GRT13, Prop. 3.8],  $\text{Psd}(P) \geq \text{Psd}(F) + 1 > \dim F + 2 = \dim P + 1$ .  $\square$

A characterization of psd-minimal polytopes in small dimensions was obtained in [GRT13] and, in particular, the following relation was shown.

**Proposition 4.8.** *Let  $V$  be a point configuration in convex position. If  $\text{Th}(V) = 1$ , then  $P = \text{conv}(V)$  is psd-minimal.*

In [GRT13] an example of a psd-minimal polytope that is not 2-level is given, showing that the condition above is sufficient but not necessary. The main result of this section is that the situation is much better for base configurations.

**Theorem 4.9.** *Let  $M$  be a matroid. The base polytope  $P_M = \text{conv}(V_M)$  is psd-minimal if and only if  $\text{Th}(M) = 1$ .*

In light of Proposition 4.8 it remains to show that there is no psd-minimal matroid  $M$  with  $\text{Th}(M) > 1$ . Since  $\mathcal{V}_{\min}^{\text{Psd}}$  is face-hereditary, it is sufficient to show that the excluded minors  $M(K_4)$ ,  $\mathcal{W}^3$ ,  $Q_6$ , and  $P_6$  are not psd-minimal.

In order to do so, we need to recall the connection to slack matrices and Hadamard square roots developed in [GRT13]. For a more coherent picture of the relations in particular to cone factorizations we refer to the papers [GPT13, GRT13]. Let  $P$  be a polytope with vertices  $v_1, \dots, v_t$  and facet-defining linear functions  $\ell_j(\mathbf{x}) = \beta - \langle a_j, \mathbf{x} \rangle$  for  $j = 1, \dots, f$ . The **slack matrix** of  $P$  is the non-negative matrix  $S_P \in \mathbb{R}^{t \times f}$  with

$$(S_P)_{ij} = \beta_j - \langle a_j, v_i \rangle$$

for  $i = 1, \dots, t$  and  $j = 1, \dots, f$ . A **Hadamard square root** of  $S_P$  is a matrix  $H \in \mathbb{R}^{t \times f}$  such that  $(S_P)_{ij} = H_{ij}^2$  for all  $i, j$ . Moreover, we define  $\text{rank}_{\sqrt{\cdot}} S_P$  as the smallest rank among all Hadamard square roots. The following is the main connection between Hadamard square roots and the psd-rank.

**Theorem 4.10** ([GRT13, Thm. 3.5]). *A polytope  $P$  is psd-minimal if and only if  $\text{rank}_{\sqrt{\cdot}}(S_P) = \dim P + 1$ .*

Thus, we will complete the proof of Theorem 4.9 by showing that the slack matrices for the excluded minors of  $\mathcal{M}_1^{\text{Th}}$  have Hadamard square roots of rank  $\geq 7$ . We start with a technical result.

**Proposition 4.11.** *The matrix*

$$A_0 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

*has  $\text{rank}_{\sqrt{\cdot}} A_0 = 4$ .*

*Proof.* Every Hadamard square root of  $A_0$  is of the form

$$H = \begin{pmatrix} 0 & y_1 & y_2 & y_3 \\ y_4 & 0 & y_5 & y_6 \\ y_7 & y_8 & 0 & y_9 \\ y_{10} & y_{11} & y_{12} & 0 \end{pmatrix}$$

with  $y_i^2 = 1$ ,  $i = 1, \dots, 12$ . Claiming that  $\text{rank}_{\sqrt{}} A_0 = 4$  is equivalent to the claim that every Hadamard square root  $H$  is non-singular. Using the computer algebra software *Macaulay2* [GS] it can be checked that the ideal

$$I = \langle y_1^2 - 1, \dots, y_{12}^2 - 1, \det H \rangle \subseteq \mathbb{C}[y_1, \dots, y_{12}]$$

contains 1 which excludes the existence of a rank-deficient Hadamard square root.  $\square$

**Proposition 4.12.** *Let  $P = P_M$  the base polytope for  $M \in \{M(K_4), \mathcal{W}^3, Q_6, P_6\}$ . Then  $\text{rank}_{\sqrt{}}(S_P) \geq 7$ .*

*Proof.* We explicitly give the argument for  $M = M(K_4)$  and  $P = P_M$ . This proof works also for the other matroids for the same choice of the collection of bases and facets. It will be sufficient to find a  $7 \times 7$ -submatrix  $A$  of  $S_P$  with  $\text{rank}_{\sqrt{}}(N) \geq 7$ . Consider the following collection of bases and facets of  $M$ :

$$\begin{array}{ll} B_1 = \{1, 2, 4\} & F_1 = \{1\} \\ B_2 = \{1, 2, 5\} & F_2 = \{2\} \\ B_3 = \{1, 2, 6\} & F_3 = \{3\} \\ B_4 = \{1, 3, 6\} & F_4 = \{4\} \\ B_5 = \{1, 4, 6\} & F_5 = \{5\} \\ B_6 = \{1, 5, 6\} & F_6 = \{6\} \\ B_7 = \{2, 4, 6\} & F_7 = \{3, 4, 6\} \end{array}$$

and the induced submatrix of  $S_P$

$$A = \begin{matrix} & \begin{matrix} \{1\} & \{2\} & \{3\} & \{4\} & \{5\} & \{6\} & \{3,4,6\} \end{matrix} \\ \begin{matrix} \{1,2,4\} \\ \{1,2,5\} \\ \{1,2,6\} \\ \{1,3,6\} \\ \{1,4,6\} \\ \{1,5,6\} \\ \{2,4,6\} \end{matrix} & \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{matrix}$$

Then  $\text{rank}_{\sqrt{}}(A) = 7$  if and only if

$$\begin{vmatrix} 0 & 0 & \pm 1 & 0 & \pm 1 & \pm 1 & \pm 1 \\ 0 & 0 & \pm 1 & \pm 1 & 0 & \pm 1 & \pm \sqrt{2} \\ 0 & 0 & \pm 1 & \pm 1 & \pm 1 & 0 & \pm 1 \\ 0 & \pm 1 & 0 & \pm 1 & \pm 1 & 0 & 0 \\ 0 & \pm 1 & \pm 1 & 0 & \pm 1 & 0 & 0 \\ 0 & \pm 1 & \pm 1 & \pm 1 & 0 & 0 & \pm 1 \\ \boxed{\pm 1} & 0 & \pm 1 & 0 & \pm 1 & 0 & 0 \end{vmatrix} = \pm \begin{vmatrix} 0 & \pm 1 & 0 & \pm 1 & \pm 1 & \pm 1 \\ 0 & \pm 1 & \pm 1 & 0 & \pm 1 & \boxed{\pm \sqrt{2}} \\ 0 & \pm 1 & \pm 1 & \pm 1 & 0 & \pm 1 \\ \pm 1 & 0 & \pm 1 & \pm 1 & 0 & 0 \\ \pm 1 & \pm 1 & 0 & \pm 1 & 0 & 0 \\ \pm 1 & \pm 1 & \pm 1 & 0 & 0 & \pm 1 \end{vmatrix} \neq 0.$$

The last determinant is of the form  $a + \sqrt{2} \cdot b$  for some integers  $a, b$ . To check that this determinant is nonzero, we can check that  $b$  is nonzero. By Laplace expansion, this is the case if

$$\begin{vmatrix} 0 & \pm 1 & 0 & \pm 1 & \boxed{\pm 1} \\ 0 & \pm 1 & \pm 1 & \pm 1 & 0 \\ \pm 1 & 0 & \pm 1 & \pm 1 & 0 \\ \pm 1 & \pm 1 & 0 & \pm 1 & 0 \\ \pm 1 & \pm 1 & \pm 1 & 0 & 0 \end{vmatrix} = \pm \begin{vmatrix} 0 & \pm 1 & \pm 1 & \pm 1 \\ \pm 1 & 0 & \pm 1 & \pm 1 \\ \pm 1 & \pm 1 & 0 & \pm 1 \\ \pm 1 & \pm 1 & \pm 1 & 0 \end{vmatrix} \neq 0.$$

The latter is exactly the claim that the matrix  $A_0$  of Proposition 4.11 has  $\text{rank}_{\sqrt{}}(A_0) = 4$ .  $\square$

## 5. HIGHER LEVEL GRAPHS

In this section we study the class  $\mathcal{G}_k^{\text{Lev}}$  of  $k$ -level graphs for arbitrary  $k$ . The Robertson-Seymour theorem assures that the list of forbidden minors characterizing  $\mathcal{G}_k^{\text{Lev}}$  is finite and we give an explicit description in the next subsection. In Section 5.2, we focus on the class of 3-level graphs which is characterized by exactly one forbidden minor, the wheel  $W_4$  with 4 spokes. The class of  $W_4$ -minor-free graphs was studied by Halin and we recover its building blocks from levelness considerations. In Section 5.3 we focus on the class of graphs with Theta rank 2. Forbidden minors for this class can be obtained from the structure of 4-level graphs.

**5.1. Excluded minors for  $k$ -level graphs.** A consequence of Theorem 3.9 is that a graph  $G$  is 2-level if and only if  $G$  does not have  $K_4$  as a minor. In order to give a characterization of  $k$ -level graphs in terms of forbidden minors, we first need to view  $K_4$  from a different angle.

**Definition 5.1.** The **cone** over a graph  $G = (V, E)$  with apex  $w \notin V$  is the graph

$$\text{cone}(G) = (V \cup \{w\}, E \cup \{wv : v \in V\}).$$

Let us denote by  $C_n$  the  $n$ -**cycle**. Thus, we can view  $K_4$  as the cone over  $C_3$ . As in the previous section, we only need to consider graphic matroids  $M(G)$  which are connected. In terms of graph theory these correspond exactly to biconnected graphs. For a facet  $F$  let us denote by  $V_F \subseteq V$  the vertices covered by  $F$ .

**Proposition 5.2.** *Let  $G = (V, E)$  be a biconnected graph and  $F \subset E$  a facet with  $|E \setminus F| \geq 2$ . Then  $G|_F$  is a vertex-induced subgraph.*

*Proof.* By contradiction, suppose that  $e \in E \setminus F$  is an edge with both endpoints in  $V_F$ . Since  $F$  is a facet,  $G/F$  is a biconnected graph with loop  $e$ . This contradicts  $|E \setminus F| \geq 2$ .  $\square$

The definition of facets requires the graph  $G/F$  to be biconnected. This, in turn, implies that  $G|_{E \setminus F}$  is connected. Let us write  $C(F) := \{uv \in E : u \in V_F, v \notin V_F\}$  for the induced **cut**. Moreover, let us write  $\bar{F} := E \setminus (F \cup C(F))$ . The next result allows us to find minors  $G'$  of  $G$  with  $\text{Lev}(G') = \text{Lev}(G)$ .

**Lemma 5.3.** *Let  $G$  be a biconnected graph and  $F$  a  $k$ -level facet. Then  $F$  is a  $k$ -level facet of the graph  $G/\bar{F}$ .*

*Proof.* Let  $H = G/\bar{F}$ . It follows from the definition of facets, that  $G|_{\bar{F}}$  is connected and thus  $H/F = G/(F \cup \bar{F}) = U_{|C(F)|,1}$  is biconnected. Moreover  $H|_F = G|_F$  is biconnected and therefore  $F$  is a facet of  $H$ .

For the levelness of  $F$ , observe that it cannot be bigger than  $k$ . Let  $T_1 \subset E$  be a spanning tree such that the restriction to the connected graph  $G|_{E \setminus F}$  is also a spanning tree. In particular,  $|T_1 \cap F|$  is minimal among all spanning trees. It now suffices to show that there is a sequence of spanning trees  $T_1, T_2, \dots, T_k \subset E$  with  $|T_i \cap F| = |T_1 \cap F| + i - 1$  for all  $i = 1, \dots, k$  and such that  $T_i \cap \bar{F} = T_j \cap \bar{F}$  for all  $i, j$ . The contractions  $T_i/\bar{F}$  then show that  $F$  is at least  $k$ -level for  $H$ .

If  $T_i \cap F$  is not a spanning tree for  $G|_F$ , then pick  $e \in F \setminus T_i$  such that  $e$  connects two connected components of  $(V_F, T_i \cap F)$ . Since  $T_i$  is a spanning tree, there is a cycle in  $T_i \cup e$  that uses at least one cut edge  $f \in C(F) \cap T_i$ . Hence  $T_{i+1} = (T_i \setminus e) \cup f$  is the new spanning tree with the desired properties.  $\square$

The contraction of  $\bar{F}$  in  $G$  gives a graph with vertices  $V_F \cup \{w\}$ , where  $w$  results from the contraction of  $\bar{F}$ .

**Proposition 5.4.** *Let  $G = (V, E)$  be a simple, biconnected graph and let  $w$  be a vertex such that the set of edges  $F$  of  $G - w$  is a facet. Then  $F$  is  $k$ -level if and only if  $\deg(w) = k$ .*



*Proof.* Let  $E_w$  be the edges incident to  $w$ . For a spanning tree  $T \subseteq E$ , we have  $\ell_F(\mathbf{1}_T) = |F \cap T| = |T| - |E_w \cap T|$ . Hence,  $F$  is  $k$ -level if and only if there are at most  $k$  spanning trees  $T_1, \dots, T_k$  such that every  $T_i$  uses a different number of edges from  $E_w$ . Since  $|E_w| = \deg(w)$  and every spanning tree contains at least one edge of  $E_w$ , there are at most  $\deg(w)$  spanning trees with different size of the intersection with  $F$ , thus  $k \leq \deg(w)$ . Moreover,  $G$  is simple, thus there exists a spanning tree  $T_1$  such that  $E_w \subseteq T_1$ . Applying the same reasoning of the proof of Lemma 5.3, we obtain the sequence of spanning trees with the desired properties. Finally, we observe that  $T_1 \cap F$  has  $\deg(w) - 1$  connected components, thus the sequence is made of at least  $\deg(w)$  trees, proving that  $\deg(w) \leq k$ .  $\square$

It follows from Proposition 5.4 that the cone over a biconnected graph on  $k$  vertices has a  $k$ -level facet. The next result gives a strong converse to this observation. A graph  $G$  is called **minimally biconnected** if  $G \setminus e$  is not biconnected for all  $e \in E$ . For more background on this class of graphs we refer to [Plu68] and [Dir67].

**Proposition 5.5.** *Let  $G$  be a simple, biconnected graph with a vertex  $w$  such that the set of edges  $F$  not incident to  $w$  is a facet. If  $F$  is  $k$ -level, then  $G$  has a minor  $\text{cone}(H)$  where  $H$  is a minimally biconnected graph on  $k$  vertices.*

*Proof.* Let  $m = |V_F|$ . By Proposition 5.4,  $\deg(w) = k$  and thus  $m \geq k$ . By removing edges if necessary, we can assume that  $F$  is minimally biconnected. By a result of Tutte (see [Oxl11, Thm. 4.3.1]) the contraction of any edge of  $F$  leaves a biconnected graph. Contract an edge such that at most one endpoint is connected to  $w$ . The new edge set  $F'$  is still a  $k$ -level facet. By iterating these deletion-contraction steps, we obtain a cone over  $F'$  with apex  $w$ .  $\square$

**Theorem 5.6.** *A graph  $G$  is  $k$ -level if and only if  $G$  has no minor  $\text{cone}(H)$  where  $H$  is a minimally biconnected graph on  $k + 1$  vertices.*

*Proof.* Let  $G = (V, E)$  be a graph and  $F \subset E$  a  $m$ -level facet such that  $m > k$ . By Lemma 5.3, we may assume that  $F$  is the set of edges not incident to some  $w \in V$ . By Proposition 5.5, we may also assume that  $G|_F$  is minimally biconnected on  $m$  vertices. Now,  $G|_F$  contains a minor  $H$  that is minimally biconnected on  $k + 1$  vertices and hence  $G$  contains  $\text{cone}(H)$  as a minor.  $\square$

**5.2. The class of 3-level graphs.** According to Theorem 5.6, the excluded minors for  $\mathcal{G}_3^{\text{lev}}$  are cones over minimally biconnected graphs on 4 vertices. The only minimally biconnected graph on 4 vertices is the 4-cycle and hence the excluded minor is  $W_4 = \text{cone}(C_4)$ , the wheel with 4 spokes. In general, let us write  $W_n = \text{cone}(C_n)$  for the  $n$ -wheel, which is a  $n$ -level graph. The family of  $W_4$ -minor-free graphs was considered by R. Halin (see [Die90, Ch. 6]). In this section, we will rediscover the *building blocks* for this class.

We start with the observation that by Lemma 3.8 and Corollary 3.5, we may restrict to 3-connected, simple graphs. Recall that a graph  $G$  (and its matroid) is  **$k$ -connected** if the removal of any  $k - 1$  vertices leaves  $G$  connected. Also, a graph is  **$k$ -regular** if every vertex is incident to exactly  $k$  edges.

**Proposition 5.7.** *A 3-level, 3-connected simple graph is 3-regular.*

*Proof.* A graph  $G$  with a vertex of degree at most 2 cannot be 3-connected. If there is a vertex  $w$  of degree at least 4, then  $G - w$  is biconnected. It follows that the set of edges  $F$  not incident to  $w$  form a facet and Proposition 5.4 yields the claim.  $\square$

The following well-known result (see [Oxl11, Thm 8.8.4]) puts strong restrictions on minimally 3-connected matroids. A  $n$ -whirl is the matroid of the  $n$ -wheel  $W_n = \text{cone}(C_n)$  with the additional basis being the rim of the wheel  $B = E(C_n)$ .

**Theorem 5.8** (Tutte's wheels and whirl theorem). *Let  $M = (E, \mathcal{B})$  be a 3-connected matroid. Then the following are equivalent:*

- (i) *For all  $e \in E$  neither  $M \setminus e$  nor  $M/e$  is 3-connected;*

(ii)  $M$  is a  $n$ -whirl or  $n$ -wheel, for some  $n$ .

We will come back to whirls in the next section. For now, we note that the only minimally 3-connected graphs are the wheels. Moreover note that every 3-regular simple graph must have an even number of vertices ( $3|V(G)| = 2|E(G)|$ ).

**Lemma 5.9.** *Let  $G$  be a 3-connected 3-regular simple graph with at least 6 vertices. Then  $G$  is at least 4-level.*

*Proof.* By assumption  $G$  cannot be a wheel. By Theorem 5.8, there must be an edge  $e$  such that  $G \setminus e$  or  $G/e$  is 3-connected. Now,  $G \setminus e$  has a degree-2 vertex for all  $e \in E$  and hence is not 3-connected. On the other hand,  $G/e$  is 3-connected and the removal of multiple edges does not alter 3-connectivity. This rules out all the cases where  $G/e$  has multiple edges, because there would be a vertex of degree 2 (not counting multiple edges). The only possibility is that  $G/e$  is a simple 3-connected graph with a vertex of degree 4. By Proposition 5.4, we conclude that  $G \setminus e$  (and consequently  $G$ ) is at least 4-level.  $\square$

**Corollary 5.10.**  $K_4$  is the only 3-level, 3-connected simple graph.

The following gives a complete characterization of level 3 graphs.

**Theorem 5.11.** *For a graph  $G$  the following are equivalent.*

- (i)  $G$  has no minor isomorphic to  $W_4$ ;
- (ii)  $G$  is 3-level;
- (iii)  $G$  can be constructed from the cycles  $C_2$ ,  $C_3$ , the dual  $C_3^*$ , and  $K_4$  by taking direct or 2-sums.

*Proof.* (i)  $\Leftrightarrow$  (ii) is Theorem 5.6 together with the fact that  $C_4$  is the unique minimally biconnected graph on 4 vertices. (ii)  $\Rightarrow$  (iii) follows from Corollary 5.10. (iii)  $\Rightarrow$  (ii) follows from Corollary 3.5 and (2.2).  $\square$

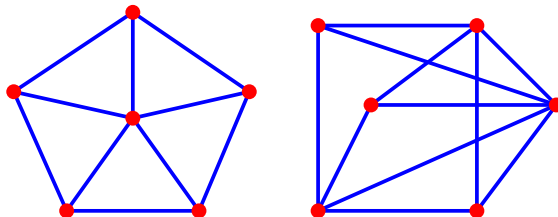
By inspecting the building blocks for 2-level (Example 7) and 3-level graphs, it is tempting to think that the building blocks of  $k$ -level graphs are given by the building blocks and the forbidden minors of  $\mathcal{G}_{k-1}^{\text{Lev}}$ . This turns out to be false even for  $\mathcal{G}_4^{\text{Lev}}$ . Indeed  $\text{Lev}(K_5) = 4$  and we cannot obtain it as a sequence of direct sums and 2-sums of  $C_2$ ,  $C_3$ ,  $C_3^*$ ,  $K_4 = W_3$ , and  $W_4$ .

**5.3. 4-level and Theta-2 graphs.** A further hope one could nourish is that 3-level graphs coincide with the graphs of Theta rank 2. This would be the case if and only if  $\text{Th}(W_4) = 3$ . The only  $k$ -level facet  $F$  of  $W_n$  with  $k > 3$  is given by the rim of the wheel  $F = E(C_n)$ . To find a sum-of-squares representation of  $\ell_F(\mathbf{x})$  for the basis configuration  $V_{M(W_n)}$  of  $W_n$ , we may project onto the coordinates of  $F$  which coincides with the configuration of *forests* of  $C_n$ . Now, every subset of  $E(C_n)$  is independent except for the complete cycle  $I = E(C_n)$ . Hence the configuration of forests is given by  $\{0, 1\}^n \setminus \{\mathbf{1}\}$  and the linear function in question is  $\ell(\mathbf{x}) = n - 1 - \sum_i x_i$ . For  $n = 4$ ,

$$18\ell(\mathbf{x}) = 2(\ell(\mathbf{x})(\ell(\mathbf{x}) - 4))^2 + (\ell(\mathbf{x})(\ell(\mathbf{x}) - 1))^2 \quad \text{for all } \mathbf{x} \in \{0, 1\}^4, \mathbf{x} \neq \mathbf{1}$$

gives a sum-of-squares representation (1.1) of degree  $\leq 2$ . We may now pullback the 2-sos representation to  $\ell_F(\mathbf{x})$  which shows that  $W_4$  is Theta-2.

Towards a list of excluded minors for  $\mathcal{G}_2^{\text{Th}}$ , we focus on the class of 4-level graphs. Using Theorem 5.6 we easily find the two excluded minors for  $\mathcal{G}_4^{\text{Lev}}$ :



The first graph is the 5-wheel  $W_5$ , the second graph is the cone over  $K_{2,3}$  and is called  $A_3 \setminus x$  in [Oxl89]. The next result states that this is the right class to study.

**Proposition 5.12.** *The wheel  $W_5$  has Theta rank 3.*

*Proof.* Let  $F = E(C_5)$  be the edges of the rim of the wheel which is a flat of rank 4. This is the unique flacet of levelness 5 and it is sufficient to show that  $4 - \ell_F(\mathbf{x})$  is not 2-sos with respect to the spanning trees  $V = V_{M(W_5)}$  of  $W_5$ . Arguing by contradiction, let us suppose that there are polynomials  $h_1(\mathbf{x}), \dots, h_m(\mathbf{x})$  of degree  $\leq 2$  such that

$$f(\mathbf{x}) := 4 - \ell_F(\mathbf{x}) - h_1(\mathbf{x})^2 - \dots - h_m(\mathbf{x})^2$$

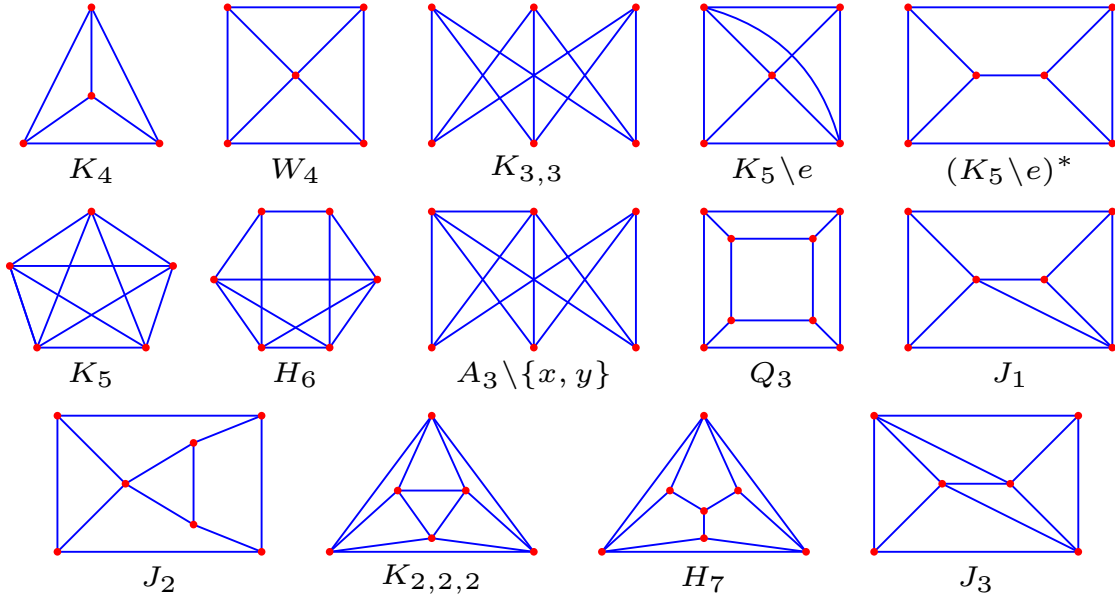
is identically zero on  $V$ .

Consider the point  $p = \mathbf{1}_F$ . This is not a basis of  $M(W_5)$  and a polynomial separating  $p$  from  $V$  is given by  $f$ . That is, by construction  $f$  is a polynomial that vanishes on  $V$  and  $f(p) \leq -1 \neq 0$ . Now we may compute a degree-compatible Gröbner basis of the vanishing ideal  $I = I(V)$  using *Macaulay2* [GS]. Evaluating the elements of the Gröbner basis at  $p$  shows that the only polynomials not vanishing on  $p$  are of degree 5. As  $\deg(f) \leq 4$  by construction, this yields a contradiction.  $\square$

The proof suggests an interesting connection to Tutte's wheels and whirls theorem (Theorem 5.8): For  $n = 4$  it states that the vanishing ideal of the  $n$ -wheel  $I(W_n)$  is generated by  $I(W^n)$  and a unique polynomial of degree  $n$ . This should be viewed in relation to Proposition 2.4: Projecting  $V_{W^n}$  and  $V_{W_n}$  onto the coordinates of  $F = E(C_n)$  yields  $\{0, 1\}^n$  and  $\{0, 1\}^n \setminus \mathbf{1}$ , respectively.

Oxley [Oxl89] determined that the class of 3-connected graphs not having  $W_5$  as a minor consists of 17 individual graphs and 4 infinite families. The graph  $A_3 \setminus x$  is clearly among these graphs and is a minor of the 4 infinite families as well as three further ones. This proves the following result.

**Theorem 5.13.** *Every 4-level graph is obtained by direct and 2-sums of  $C_2$ ,  $C_3$ ,  $C_3^*$ , and the following 14 graphs*



$\square$

As  $A_3 \setminus x$  is Theta-2, a complete list of excluded minor has to be extracted from the 17 graphs plus 4 families in [Oxl89]. As a last remark, we note that the Theta-1 graphs are given by series-parallel graphs. The property of being Theta-2 however is independent of planarity.

**Proposition 5.14.** *The graphs  $K_5$  and  $K_{3,3}$  have Theta rank 2.*

*Proof.* For both cases we use the idea that for a given facet  $F \subseteq E$ , we may project the basis configuration  $V$  onto the coordinates given by  $F$  and find a 2-sos representation of the linear function  $\text{rank}(F) - \sum_i x_i$ .

For the graph  $K_{3,3}$ , the only facets of levelness  $> 3$  are given by 4-cycles. Projecting onto these coordinates yields  $\{0, 1\}^4 \setminus \mathbf{1}$  which is a point configuration of Theta rank 2.

For the complete graph  $K_5$ , we note that the only facets  $F$  of levelness  $> 3$  are given by the edges of an embedded  $K_4$ . For such a facet, we might equivalently consider  $\ell_{E \setminus F}(\mathbf{x}) - 1 \geq 0$ . Projecting onto  $E \setminus F$  again yields  $\{0, 1\}^4 \setminus \mathbf{0}$ .  $\square$

## 6. EXCLUDED MINORS FOR $k$ -LEVEL MATROIDS

The cone construction (Definition 5.1) employed in the previous section to show the existence of finitely many excluded minors for  $\mathcal{G}_k^{\text{lev}}$  cannot be extended to general matroids. Indeed, whereas any two trees on  $n$  vertices have the same matroid, the matroid of their cones typically do not. Moreover, there is no Robertson-Seymour theorem for general matroids: Minor-closed classes of matroids are generally not characterized by finitely many excluded minors. In this section we show that  $k$ -level matroids can be characterized in finite terms and we describe the class explicitly.

A matroid  $M$  is called **minimally  $k$ -level** if  $\text{Lev}(M) = k$  and  $\text{Lev}(N) < \text{Lev}(M)$  for every minor  $N$  of  $M$ . It is clear that excluded minors for  $\mathcal{M}_k^{\text{lev}}$  are given by the minimally  $l$ -level matroids for  $l > k$ . The main result of this section is the following.

**Theorem 6.1.** *Excluded minors for the class of  $(k-1)$ -level matroids are given by the minimally  $k$ -level matroids. In particular, the list of excluded minors for  $\mathcal{M}_{k-1}^{\text{lev}}$  is finite.*

Let us formalize a notion that we already saw in the proof of Lemma 5.3: A  **$k$ -sequence of bases** for a facet  $F$  is a collection of bases  $B_1, \dots, B_k \in \mathcal{B}(M)$  such that

- $|F \cap B_1|$  is minimal among all bases of  $M$ ,
- $|F \cap B_{i+1}| = |F \cap B_i| + 1$ , for  $1 \leq i < k$ ,
- $F \cap B_i \subset F \cap B_{i+1}$ , for  $1 \leq i \leq k-1$ , and
- $|F \cap B_k| = \text{rank}_M(F)$ .

It is straightforward to verify that a facet  $F$  is  $k$ -level if and only if  $F$  has a  $k$ -sequence of bases. Indeed, starting with a basis  $B_1$  such that  $|F \cap B_1|$  is minimal, one iteratively alters  $B_{i+1}$  by some  $e_i \in F \setminus B_i$ . We can also make a more refined choice.

**Lemma 6.2.** *Let  $M$  be a connected matroid and  $F$  a  $k$ -level facet. For any  $e \in \overline{F} := E(M) \setminus F$ , there exists a  $k$ -sequence of bases  $B_1, \dots, B_k$  such that  $e \in B_i$  for  $i = 1, \dots, k$ .*

*Proof.* Since  $M$  is connected,  $e$  is not a loop and we can find a basis  $B_1$  such that  $|F \cap B_1|$  is minimal and  $e \in B_1$ . For  $1 \leq i < k$ ,  $|F \cap B_i| < \text{rank}(F)$ . So, there is an  $e_i \in F \setminus B_i$  such that  $(F \cap B_i) \cup \{e_i\}$  is independent. Let  $C_i \subseteq B_i \cup \{e_i\}$  be the fundamental circuit containing  $e_i$ . Since  $F$  is a flat and  $C_i$  is not a circuit in  $F$ , there is  $f_i \in C_i \setminus (F \cup \{e_i\})$  and we define  $B_{i+1} = (B_i \setminus f_i) \cup e_i$ .  $\square$

Since  $M_1 \oplus_2 M_2$  contains both  $M_1$  and  $M_2$  as minors, it follows from Lemma 3.8 and Corollary 3.5 that every minimally  $k$ -level matroid is 3-connected.

**Proposition 6.3.** *Let  $M$  be a minimally  $k$ -level matroid and  $F$  a  $k$ -level facet of  $M$ . Then  $(M/F)^*$  is a minimally connected matroid.*

*Proof.* Suppose  $(M/F)^*$  is not minimally connected. There exists an element  $e \in \overline{F}$  such that the deletion  $(M/F)^* \setminus e$  is a connected matroid. Since a matroid is connected if and only if its dual is, we infer that  $(M/F)/e$  is connected.

Since  $M$  is minimally  $k$ -level, it is 3-connected. By Lemma 6.2 we can construct a  $k$ -sequence of bases for  $F$  such that all bases contain  $e$ . We have that  $B_1 \setminus e, \dots, B_k \setminus e$  is a  $k$ -sequence of

bases for  $F$  with respect to the matroid  $M/e$ . We only need to check that  $F$  is a facet of  $M/e$ . If  $C$  is a circuit containing  $e$  and some elements of  $F$ , it must contain at least a second element  $e' \in \overline{F}$  because  $F$  is a flat. In addition, there must be at least a third element  $e'' \in \overline{F}$ , otherwise  $e'$  would be a loop of  $(M/F)/e$ , which is connected by hypothesis. This shows that  $F$  is a flat of  $M/e$ . Moreover,  $(M/e)/F \cong (M/F)/e$  and  $(M/e)|_F \cong M|_F$  are connected. Thus  $F$  is a  $k$ -level facet of  $M/e$ , contradicting the  $k$ -level minimality of  $M$ .  $\square$

Similar to the case of graphs, the following proposition states that  $\overline{F} = E(M) \setminus F$  is independent for a  $k$ -level facet of a minimally  $k$ -level matroid.

**Proposition 6.4.** *Let  $M$  be a minimally  $k$ -level matroid and  $F$  a  $k$ -level facet of  $M$ . Then  $\text{rank}(\overline{F}) = |\overline{F}|$ .*

*Proof.* By contradiction, suppose  $\text{rank}(\overline{F}) < |\overline{F}|$ . Consider a  $k$ -sequence of bases  $B_1, \dots, B_k$  for  $F$ . Because of the assumption  $\text{rank}(\overline{F}) < |\overline{F}|$ , we can pick an element  $e \in \overline{F} \setminus B_1$ . By Proposition 6.3,  $(M/F)/e$  is not connected. Since  $F$  is a facet,  $M/F$  is connected and, by [Oxl11, Thm. 4.3.1],  $(M/F) \setminus e$  is connected. Now  $F$  is a flat of  $M \setminus e$  and both  $(M \setminus e)|_F \cong M|_F$  and  $(M \setminus e)/F \cong (M/F) \setminus e$  are connected. Hence,  $F$  is a facet of the matroid  $M \setminus e$ . The bases  $B_1, \dots, B_k$  are also bases for  $M \setminus e$  and form a  $k$ -sequence for the facet  $F$ . Thus  $M \setminus e$  is a  $k$ -level minor of  $M$ , contradicting the  $k$ -level minimality of  $M$ .  $\square$

**Proposition 6.5.** *Let  $M$  be a minimally  $k$ -level matroid and  $F$  a  $k$ -level facet of  $M$ . Then  $M|_F$  is a minimally connected matroid.*

*Proof.* Suppose that  $(M|_F) \setminus e$  is connected for some  $e \in F$ . Then  $\hat{F} = F \setminus e$  is a flat of  $M \setminus e$ . We show that  $\hat{F}$  is a  $k$ -level facet of  $M \setminus e$ .

The matroid  $(M \setminus e)|_{\hat{F}} \cong (M|_F) \setminus e$  is connected by hypothesis. Note that  $M/\hat{F}$  has  $e$  as a loop and hence  $(M \setminus e)/\hat{F} \cong M/F$ . Thus,  $(M \setminus e)/\hat{F}$  is also connected which shows that  $\hat{F}$  is a facet.

At last, we show that there is a  $k$ -sequence of bases of  $M \setminus e$  for  $\hat{F}$ . Since  $M|_F$  is connected it has a basis that avoids  $e$ . We can complete this to a basis  $B_k$  of  $M$ . Now for  $f \in \overline{F}$ ,  $B_k \cup f$  contains a circuit and by Proposition 6.4 this circuit is not entirely in  $\overline{F}$ . Hence, we can define a basis  $B_{k-1} := B_k \setminus f \cup f'$  for some  $f' \in B_k \cap F$ . Continuing this way yields a  $k$ -sequence of bases  $B_1, \dots, B_k$  for  $F$  that avoids  $e$  and hence is a  $k$ -sequence of bases for  $\hat{F}$  in  $M \setminus e$ . This contradicts the  $k$ -level minimality of  $M$ .  $\square$

**Proposition 6.6.** *Let  $M$  be a minimally  $k$ -level matroid and  $F$  a  $k$ -level facet of  $M$ . Then  $\text{rank}(F) = k - 1$ .*

*Proof.* Suppose that  $\text{rank}(F) > k - 1$ . Consider a  $k$ -sequence  $B_1, \dots, B_k$  for  $F$ : by definition  $|F \cap B_k| = \text{rank}(F) > k - 1$  and thus  $|F \cap B_1| > 0$ . Equivalently, there is an element  $e \in F$  such that  $e \in B_i$  for  $i = 1, \dots, k$ . We prove that the matroid  $M/e$  is  $k$ -level with respect to the facet  $\hat{F} = F \setminus e$ .  $(M/e)|_{\hat{F}} \cong (M|_F)/e$  is connected because  $M|_F$  is minimally connected by Proposition 6.5 and  $(M/e)/\hat{F} \cong M/F$  is connected because  $F$  is a facet of  $M$ . Finally,  $B_1 \setminus e, \dots, B_k \setminus e$  are bases of  $M/e$  and form a  $k$ -sequence for the facet  $\hat{F}$ , contradicting the  $k$ -level minimality of  $M$ .  $\square$

We can finally show that the excluded minors of  $\mathcal{M}_k^{\text{lev}}$  are given by the minimally  $(k + 1)$ -level matroids.

**Proposition 6.7.** *Every minimally  $(k + 1)$ -level matroid has a  $k$ -level minor.*

*Proof.* Let  $M$  be a minimally  $(k + 1)$ -level matroid and  $F$  a  $(k + 1)$ -level facet. Choose a  $k$ -sequence  $B_0, \dots, B_k$  for  $F$ . Pick an element  $f \in F \setminus B_1$  such that  $f \in B_i$  for  $i = 1, \dots, k$ . Applying the same reasoning as in the proof of Proposition 6.6, we infer that  $\hat{F} := F \setminus f$  is a facet of  $M/f$ . Moreover,  $B_1 \setminus f, \dots, B_k \setminus f$  is a  $k$ -sequence of bases which shows that  $M/f$  is  $k$ -level.  $\square$

To complete the proof of Theorem 6.1, we show that for fixed  $k$ , the size of the ground set of a minimally  $k$ -level matroid is bounded. This trivially implies that there only finitely many minimally  $k$ -level matroids. To bound the size of the ground set of a minimally  $k$ -level matroid  $M$ , we choose one of its  $k$ -level flacets  $F$  and bound separately the size of  $F$  and the size of its complement  $\bar{F} = E(M) \setminus F$ . We quote two useful facts from Oxley's book.

**Proposition 6.8.** [Oxl11, Prop. 4.3.11] *Let  $M$  be a minimally connected matroid of rank  $r$  where  $r \geq 3$ . Then  $|E(M)| \leq 2r-2$ . Moreover, equality holds if and only if  $M \cong M(K_{2,r-1})$ .*

**Proposition 6.9.** [Oxl11, Ch. 4, Ex. 10 (d)] *Let  $M$  be a matroid for which  $M^*$  is minimally connected. Then either  $M \cong U_{n,1}$  for some  $n \geq 3$  or  $M$  has at least  $\text{rank}(M)+1$  non-trivial parallel classes.*

*Proof of Theorem 6.1.* In light of Theorem 3.9, we only need to consider  $k \geq 3$ . Let  $M$  be a minimally  $k$ -level matroid  $M$ . Any  $k$ -level flacet  $F$  of  $M$  is of rank  $k-1$  by Proposition 6.6; By Proposition 6.5,  $M|_F$  is minimally connected. If  $\text{rank}(F) = 2$ , then Proposition 6.9 implies that  $M|_F \cong U_{3,2}$ . For  $\text{rank}(F) \geq 3$ , by Proposition 6.8,  $F$  has at most  $2(k-1)-2 = 2k-4$  elements. Hence, we need to upper bound the number of elements in  $\bar{F}$ . Set

$$T := \{e \in \bar{F} : \exists C \text{ circuit of } M \text{ with } e \in C \text{ and } |C \cap \bar{F}| = 2\}.$$

That is, every  $e \in T$  is in a non-trivial parallel class in  $M/F$ . The number of non-trivial parallel classes is bounded from above by  $\frac{|T|}{2}$ . Set  $S := \bar{F} \setminus T$ .

Define  $h := \text{rank}(M) - \text{rank}(F) - 1$ , so that  $\text{rank}(M/F) = h+1$ . By Proposition 6.3,  $(M/F)^*$  is minimally matroid on at least 3 elements (since for  $k \geq 3$  this implies  $|\bar{F}| \geq 4$ ). By Proposition 6.9 there are two possibilities:

If  $M/F \cong U_{|\bar{F}|,1}$ , then  $\text{rank}(M) = k$  and  $|\bar{F}| \leq k$  because of Proposition 6.4. It follows that  $|E(M)| = |F| + |\bar{F}| \leq 2k-4+k = 3k-4$ . If  $k = 2$ , then  $|\bar{F}| \leq 3$ .

On the other hand, if  $\text{rank}(M/F) = h+1 > 1$ , then  $M/F$  has at least  $h+2$  non trivial parallel classes. Hence we obtain  $|T| \geq 2h+4$ . Moreover,  $\bar{F}$  has exactly  $\text{rank}(F) + h + 1 = k + h$  elements and this fact yields  $|T| \leq k + h$ . Together this gives

$$2h+4 \leq k+h \implies h \leq k-4.$$

It is immediate that  $|\bar{F}| = k+h \leq 2k-4$  and finally  $|E(M)| = |F| + |\bar{F}| \leq 2k-4+2k-4 = 4k-8$ .  $\square$

The result of this section does not rule out that matroids of Theta rank  $k$  have infinitely many excluded minors and we did not manage to extend our techniques to Theta rank. However, we conjecture that the class  $\mathcal{M}_k^{\text{Th}}$  of matroids of Theta rank  $k$  is described by finitely many excluded minors.

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